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Quantum Cosmology

Cosmology is normally regarded as classical physics. When Stephen was at IoA his work on cosmology was purely classical; as a result of that work and Penrose's it was possible to show that classical relativity predicted that there would be an initial singularity - the classical theory is predicting its own breakdown. This is like the classical theory of the atom at the beginning of the century, when the picture of a classical electron orbiting a nucleus predicted that the electron would radiate energy and spiral into the nucleus, so all matter would collapse. This motivated the development of quantum mechanics. SWH will show that in GR the problem of the classical singularity can also be overcome in QM but before he gets onto that - which is a very speculative area - he's going to deal with a more established area, quantum field theory in curved spacetime.

In the early universe, when the radius of curvature was somewhat bigger than the Planck length, it ought to be a good approximation to neglect quantum fluctuations of the metric, but even so quantum effects are important for the matter fields. Stephen will deal with the inflationary universe model.

We take a different approach to QFT from that normally given in lectures - the path integral approach. The reason is that this can be carried over very nicely to the case of gravity. We will start with the case of ordinary quantum fields in flat spacetime.

$-+++$ metric

Simplest possible case: scalar field.

The (complex) "probability" that the field is in a given configuration $\phi(x,t)$ is $\propto e^{iI[\phi]}$
where the action I is

$$I[\phi] = \frac{1}{2}\int -(\nabla\phi)^2 - m^2\phi^2 \, d_3x\,dt$$

We now go from Minkowski space to Euclidean space using $\tau = it$; we now have a $++++$ metric.

Probability of $\phi(x,\tau) \propto e^{-\hat{I}[\phi]}$
where

$$\hat{I}[\phi] = -iI[\phi] = \frac{1}{2}\int (\nabla\phi)^2 + m^2\phi^2 \, d_3x\,d\tau \geq 0$$

$\hat{I}$ is positive definite and minimized by $\phi = 0$ which must therefore correspond to the most probable configuration.
The further you get away from $\phi = 0$, the lower is the probability of finding the field in that configuration. The physics of the system is defined by probabilities of this form for all configurations $\phi(x,\tau)$ which belong to a certain class C.

Now go back to real spacetime. The physics is defined by the pseudo-probabilities e^{+iI} for all field configurations belonging to a certain class C in the infinite past and to the complex conjugate class C^* in the inf. future.

Suppose we want the field ϕ to be in its vacuum state.
A field of energy E has time dependence $e^{iEt} = e^{-E\tau}$
So the fields blow up as $\tau \to -\infty$. To have a zero energy state require that fields $\in C$ are regular as $\tau \to -\infty$
fields $\in C^*$ " " " $\tau \to +\infty$

The vacuum state has fields $\phi \in C \cap C^*$,
it is defined by probabilities of the form $e^{-\hat{I}[\phi]}$ for all configurations regular at $\tau = \pm\infty$

We are not interested in the probability of a whole field configuration, but in a more restricted quantity, the values of ϕ at n events (x_i, τ_i); for instance.

$$\langle 0|\phi|0\rangle \equiv \langle 0|\phi(x_1,\tau_1)\phi(x_2,\tau_2)\ldots\phi(x_n,\tau_n)|0\rangle$$

This may be calculated in a straightforward way.

$$\langle 0|\phi|0\rangle = \frac{\int_{C\cap C^*} d[\phi]\ \phi(x_1,\tau_1)\ldots\phi(x_n,\tau_n)\ e^{-\hat{I}[\phi]}}{\int_{C\cap C^*} d[\phi]\ \ e^{-\hat{I}[\phi]}}$$

$d[\phi]$ is a measure over the ∞ dimensional space of all field configurations. In order to calculate this correlation function we add a source term J to the action:

$$\hat{I}[\phi, J] = \frac{1}{2}\int (\nabla\phi)^2 + m^2\phi^2 + J(x,\tau)\phi(x,\tau)\ d_3x\,d\tau$$

Let $$Z[J] = \int e^{-\hat{I}[\phi,J]} d[\phi]$$

Functionally differentiate Z wrt source term J at the various points $x_i\tau_i$

$$\langle 0|\phi|0\rangle = \left[\frac{(-1)^n}{Z[J]}\frac{\delta^n Z}{\delta J(x_1,\tau_1)\ldots\delta J(x_n,\tau_n)}\right]_{J=0}$$

Having evaluated this in Euclidean space, then analytically continue in the variables τ_i back to real values of t_i. Get the expectation value corresponding to this particular operator ordering $\phi(x_1,t_1)\ldots\phi(x_n,t_n)$ in real Minkowski spacetime.

Note that we have used the fact that the metric is positive definite but not that it is flat - we could equally well have used a curved metric.

Define $W = -\ln Z$,
Then

$$<0|\phi|0>_J = \frac{\delta W}{\delta J(x,\tau)},$$

the expectation value of ϕ in the presence of a source term J.
You can regard this as an equation which determines J given $<0|\phi|0>$.
A Legendre transformation then gives the action

$$\Gamma[<\phi>] = W[J] - \int J(x,\tau)<\phi>\, d_3x\, d\tau$$

$$\delta\Gamma = \frac{\delta W}{\delta J}\delta J - \int \delta J <\phi> d_3x\, d\tau - \int J\delta<\phi> d_3x\, d\tau$$

but $\left(\frac{\delta W}{\delta J} - <\phi>\right)\delta J = 0$

so $\delta\Gamma = -\int J\delta<\phi>\, d_3x\, d\tau$

$$\frac{\delta\Gamma}{\delta<\phi>} = -J$$

so if $J=0$ (no source) the field equation is

$$\frac{\delta\Gamma}{\delta<\phi>} = 0$$

Γ will be a very complicated (and in general non local) function of $<\phi>$
However, a simple case is

$<\phi> = \text{const}$

$\Gamma = \int V(<\phi>)\, d_3x\, d\tau$, V is the 'effective potential' of the field theory.

This quantity V will play an important role in the next lecture when Stephen will deal with the inflationary Universe.

(At this point Stephen's chair fell back off the dais and after being rescued joked that he had fallen off the edge of the universe.)

The first case of a solution of the Einstein equations where these ideas of a positive definite metric are of use is just the Schwarzschild metric

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

Write it $= \tau$, then

$$ds^2 = \left(1-\frac{2m}{r}\right)d\tau^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

The apparent singularity at $r=2m$ is removable by a coordinate transformation

$$x = 4M\left(1-\frac{2m}{r}\right)^{1/2}$$

so

$$ds^2 = x^2\left(\frac{d\tau}{4m}\right)^2 + \left(\frac{r}{2m}\right)^4 dx^2 + r^2 d\Omega^2$$

(This metric is like plane polars

$$ds^2 = \rho^2 d\theta^2 + d\rho^2$$

Especially near $r=2m$ where the quantity in front of dx^2 is unity. Identify angular coordinate θ with $\tau/4m$ to make the metric regular at $r=2M$; τ must then be periodic of period $8\pi M$.

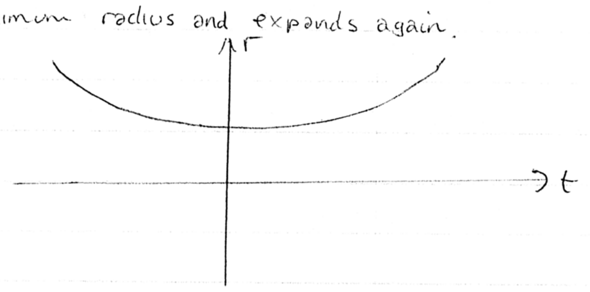
The quantum ~~fields~~ states given by fields which are regular on a space periodic in ~~with~~ imaginary time with a period ~~4π~~ β, ~~then~~ are thermal states with a temperature of $1/\beta$.

This is why black holes have temperature $T = 1/8\pi M$

A metric of much greater interest to us is that of de Sitter space, which can be written in various RW forms.
The $k = +1$ form covers the whole ST.

$$ds^2 = -dt^2 + \frac{1}{H^2} \cosh Ht \, d\Omega_3^2$$

$d\Omega_3^2$ is the metric on a 3-sphere of unit radius
This represents a closed universe which collapses from ∞, goes down to minimum radius and expands again.



$H^2 = \Lambda/3$ $\qquad \tau = it$

$$ds^2 = d\tau^2 + \frac{1}{H^2} \cos H t \, d\Omega_3^2$$

This is just the metric on a 4-sphere with $\tau = 0$ at the equator
$\tau = \pi/2H$ @ N pole
$\tau = -\pi/2H$ @ S pole

This is a positive definite metric which is compact.
If you look at the orbits of τ; they are great circles
The metric is periodic in the imaginary time coordinate with period $\frac{2\pi}{H}$.

Hence a field theory can be defined in de Sitter space with a natural temperature $H/2\pi$ $\quad (H\hbar/k)$
$\frac{kT}{\hbar} = H$.

Did the Universe have a de Sitter-like period in its early stages?
Stephen will describe how this can account for some unexplained features of the standard hot big bang model.

The Inflationary Universe.

This idea has gained much popularity in the past few years. It is probably incorrect as it stands, but some of the ideas are right, and they can probably be applied to inflation at a much earlier era, maybe even the Planck era. I will come to that later on, but it is helpful to start by dealing with the more conventional GUT era first.

The hot big bang theory of the Universe has had great success in describing observations like the microwave background and the abundances of light elements. I briefly summarize hot big bang theory, which assumes that the universe is spatially homogeneous and isotropic and has a metric of one of the Robertson - Walker models

$$ds^2 = -dt^2 + R^2 \left[\frac{dr^2}{1+kr^2} + r^2 d\Omega^2 \right]$$

where $k=+1$ closed, recollapsing model

$k=0$ flat, ~~open~~ spatial sections ; open

$k=-1$ open model.

In this model, $R(t)=0$ at $t=0$, when there is thermodynamic equilibrium at infinite temperature. The universe then expands and the temperature drops. If T is the temperature, ρ the energy density, and N the number of effectively zero mass particles' spin states,

$$\rho = \frac{\pi^2}{90} N T^4$$

With that equation of state $T \sim 1/R$ $R \sim t^{1/2}$

At $t = 1$ second, $T \sim 10^{10}$ K. This is when you start the formation of light elements like helium. After 10^5 or 10^6 years T falls to a few thousand K, and electrons start recombining with protons to form hydrogen. This is also about the time when the universe ceases to be dominated by relativistic particles and becomes dominated by particles with rest mass. After this

$$\rho \sim R^{-3} \qquad R \sim t^{2/3}$$

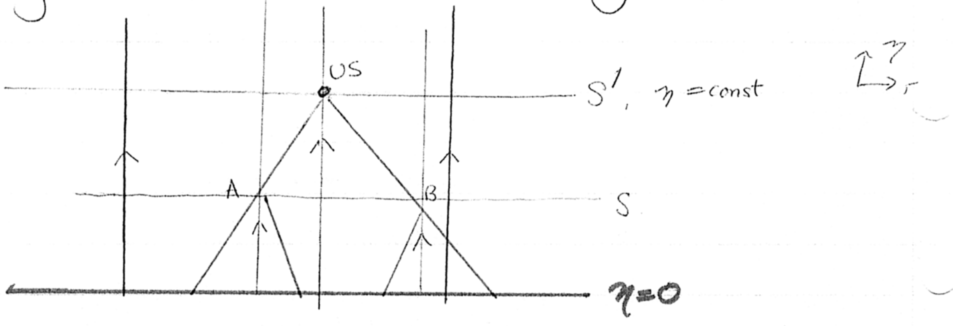
It is convenient to introduce a new time variable η,

$$d\eta = \frac{dt}{R}$$

so

$$ds^2 = R^2\left(-d\eta^2 + \frac{dr^2}{1+kr^2} + r^2 d\Omega^2\right)$$

At early times you can neglect the effect of the kr^2 term so the metric is just that of flat space.
The world lines of particles are vertical lines in η, r space and light cones are at 45°; this is very useful later on.



[At the point on their world lines where A and B lie on our past light cone, they had never been in causal contact].

The hot big bang model agrees with observations but isn't completely satisfactory because a number of important features are assumed as initial conditions rather than explained.

(1) The Horizon Problem

Light can't propagate freely along our past light cone from $\eta = 0$ - rather a pity, otherwise we could see back to the beginning. At early times the matter of the universe was highly ionized, so there were a lot of electrons to scatter the light. These electrons disappeared when the temperature fell to a few thousand degrees because they then combined with protons to form hydrogen. There is a surface S where light was last scattered. After this time, light would have propagated freely; when we look back we find we are receiving microwaves at a temperature of about $3^\circ K$ which must have been last scattered on S. The really remarkable feature is that the temperature is always exactly the same no matter which direction we look in, to 1 part in 10^3 or 10^4. That is really remarkable because if we look in different directions toward A and B we are looking at regions which were never able to communicate through their past light cone. If they had a common region in the past we might say that it's because of some equilibrium process in that region, ~~tha~~ but if they were never in contact how on earth do they know to have the same temperature? A big bang universe like this is said to have a particle horizon. All this means is that the pasts of different points do not intersect.

② <u>The Flatness problem.</u>

Is our universe closed or open? We don't know whether it will recollapse or not. We can determine whether or not this will happen by finding whether the density of the universe is higher or lower than for the $k=0$ model. in fact we can define a parameter Ω which is the ratio of the actual density of the universe to the ~~initial~~ critical density ρ_c. Adding up all the luminous mass we can observe, we find $\Omega_{lum} \sim 0.01$. There is fairly good evidence that there must be quite a lot of dark matter around in order to make galaxies and clusters of galaxies gravitationally bound, say $\Omega \sim 0.1$. There might be quite a lot of matter distributed uniformly throughout the universe, and it's quite possible that $\Omega = 1$. Even if Ω is only 0.1, or 0.01, this is still quite remarkable, as $\Omega=1$ is a highly unstable state: if at one time the universe departs slightly from $\Omega=1$ then at later times it will depart much further. $\Omega = 0.1$ now means $|\Omega - 1| < 10^{-19}$ at 1 second. How did the universe 'know' that it had to start with Ω so very nearly equal to 1?

③ <u>The smoothness problem.</u>

If the universe really was ~~flat~~ hot at early times you'd expect there to be fluctuations from statistical equilibrium, with fluctuations of order $N^{1/2}$ in Ω for every N particles. The number of particles is frozen in by the number coming out of the initial singularity; since the different regions were not in causal contact, the $N^{1/2}$ fluctuations were frozen in.

We'd expect fluctuations in the microwave background much bigger than the observed limits of 10^{-4}.

(4) The Monopole problem.

I don't want to spend much time on this as it's quite possible that monopoles don't exist. GUTs predict: 10^{15} GeV (10^{-7} g) monopoles (fairly light, but heavy by elementary particle standards!) In GUTs you'd expect monopoles to be formed when the temperature fell below about 10^{15} GeV. The direction of symmetry breaking would be different in different regions. At later times, as these regions began to see each other, they would try to line up their direction of symmetry breaking, but they wouldn't always be able to do this because they'd find that in some directions they'd tied themselves up in knots which are just the monopoles. The monopoles are knots in the direction of symmetry breaking. The number of them that you would expect is bigger than observed limits by a factor of 10^{14} - something is wrong with hot big bang or grand unification or both.

(5) The Baryon Problem.

There is fairly good evidence that our galaxy is composed entirely of baryons, with ~~a~~ a completely insignificant amount of antibaryons. We can't be quite so sure about other galaxies - Andromeda might be an antigalaxy. But if there were an equal number of galaxies and antigalaxies we'd expect there to be regions where matter and antimatter annihilated rather ~~then~~ frequently, producing lots of γ rays which we don't detect, so all the observed universe is made of baryons and not antibaryons. On this assumption the baryon-photon ratio is about $10^{-9\pm1}$. This number is a constant in the hot big bang theory and has to be put in as an initial condition, it cannot be explained.

⑥ The cosmological constant problem.

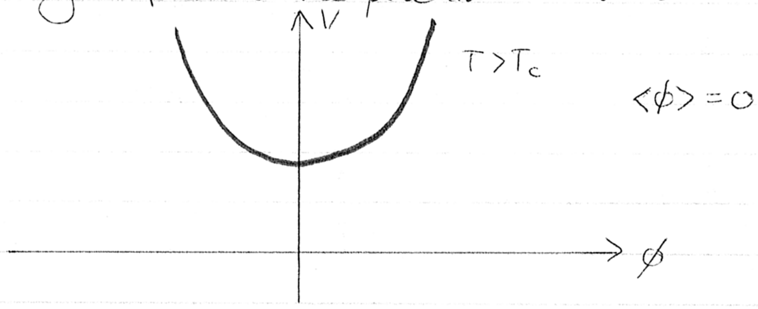
By looking at distant galaxies we can measure $H_0 = \dot{R}/R$ and the deceleration parameter $q_0 = -\ddot{R}/RH_0^2$. If one knew the matter density and could measure q_0, one could find Λ. $|q_0| < 5$ gives an upper limit $|\Lambda|/M_{pl}^2 < 10^{-120}$, M_{pl} the Planck mass. In fact this makes Λ the quantity in physics which is most accurately measured, however there are a number of reasons from quantum theory which might lead us to expect Λ to be much higher, even of order 1.

The inflationary universe doesn't solve the cosmological constant problem - it assumes $\Lambda = 0$. Stephen will be giving a talk in the lunchtime seminar series about a possible explanation for Λ being 0 and doesn't want to go into that now.

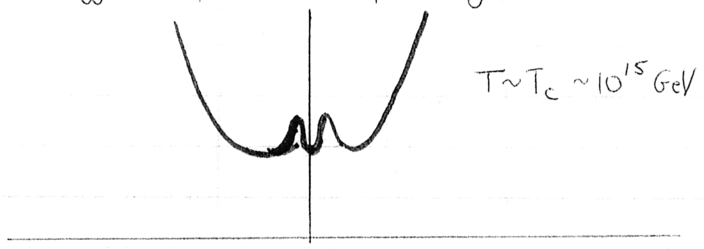
If we believe GUTs, there is a possibility that baryon non-conserving processes happen at very high energies, and if we believe the hot big bang, there were very high energies at early times. We also need some reason for getting baryons and not antibaryons. In general, we believe QFT is invariant under CPT, charge conjugation, parity and time reversal. The early universe is highly non-T invariant because of the expansion and cooling, so if particle interactions violate CP in this non time symmetric situation, you can end up with an excess of particles rather than antiparticles, even if you start off with equal numbers.

We have no reason to predict that we get baryons rather than antibaryons, but if the universe turned out to be made of antibaryons we'd just have switched the labels around. Taking a particular theory and putting in a particular amount of CP violation you can calculate the ratio of baryons to photons

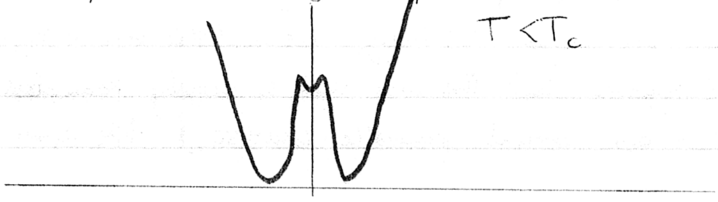
and you can get any answer you want up to about 10^{-4}, so we can explain the observed 10^{-8}, but we cannot claim that it is predicted. This leaves problems 1 - 4. These are the problems that the inflationary scenario tries to explain.
The idea is that at very early times, the temperature is very high. These inflationary models in their original form were proposed in the context of grand unified theories, where the idea is that the strong interaction, weak interaction and electromagnetism are unified at energies of about 10^{15} GeV and are really all the same at energies higher than this, but at energies lower than this, there is spontaneous symmetry breaking which is caused by a Higgs scalar field which breaks the symmetry between the strong and electroweak interactions. There is some gauge group G (eg $SU(5)$, $SO(10)$, ...) and a Higgs scalar ϕ. The Universe starts off at the Planck temperature, 10^{19} GeV.
At that high temperature the potential looks like



As the universe expands, the temperature drops and the form of the effective potential ~~drops~~ changes



And, at some temperature T_c, develops minima at nonzero $\langle\phi\rangle$, but the potential kept at $\langle\phi\rangle = 0$ by the potential barriers.



As the universe cools, the minimum at $\phi = 0$ is not the lowest energy state. There are other minima at nonzero $\langle \phi \rangle$, but ϕ is still trapped by the barriers. As the universe continues to expand the temperature drops and the energy density is made up of 2 terms:

$$\rho \sim aT^4 + V(0)$$

The thermal term is dominant at early times but is very small at later times where the dominant term is $V(0)$. Early on, $\rho \sim R^{-4}$; the field equations give $R \sim t^{1/2}$. When $V(0)$ dominates, $R \sim e^{Ht}$ so get exponential expansion.

$$H^2 = \frac{8\pi V(0)}{3m_p^2}$$

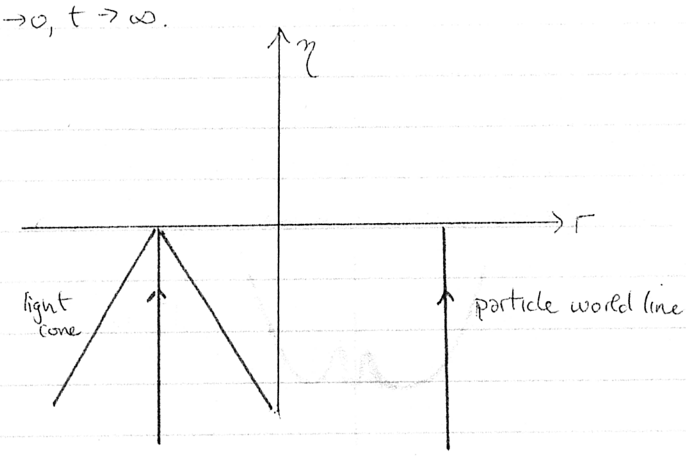
Using conformal time $d\eta = dt/R$,

$$ds^2 = R^2(-d\eta^2 + d\Omega_3^2)$$

so

$$\eta = -\frac{1}{H}e^{-Ht}, \qquad R \sim \frac{1}{\eta}$$

As $\eta \to 0$, $t \to \infty$.



In this universe the past light cones of any two events X, Y intersect. In the hot big bang model the light cones were cut off and didn't all intersect. (Strictly, we will still have a singularity at η_0 far and negative, but the area that is causally connected is much larger than our present observable universe.) We don't want the universe

At early times the potential would also have another term which was like $\phi^2 T^2$:

$$V = \ldots + \phi^2(m^2 + \xi R) + D\phi^2 T^2 + V_0$$

Now at early times this keeps the expectation value of ϕ zero but as the universe exponentially expanded the temperature drops exponentially and thus term goes away. If suppose $m^2 + \xi R = 0$, get the new form of the potential. So the $\phi = 0$ position is now unstable, and ϕ begins to roll down the hill. Because the potential is very flat at the top it rolls very slowly.

The field obeys the equation

$$\Box\phi = -\frac{\partial V}{\partial \phi}$$

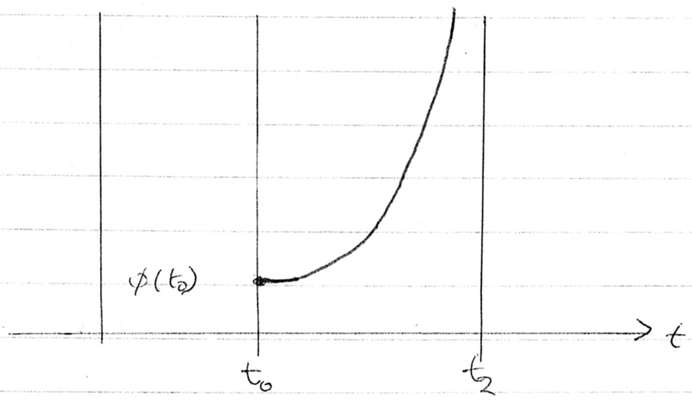
$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial \phi}$$

$$\sim \lambda\phi^3,$$

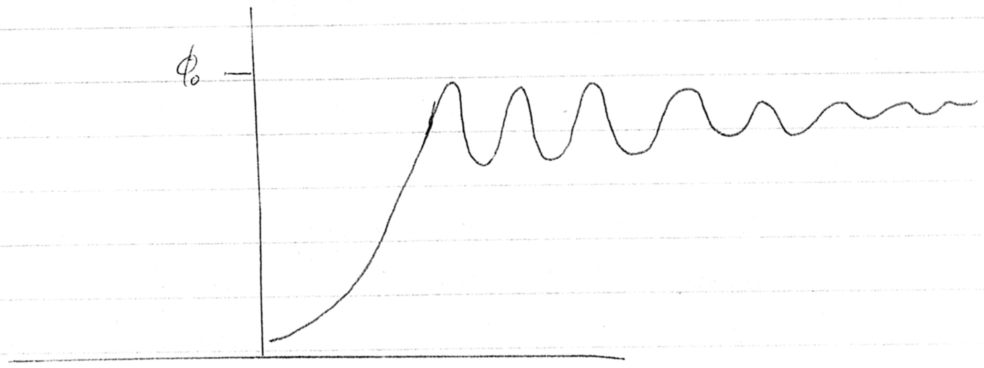
(since $\frac{\partial V}{\partial \phi} \approx 4ce^4\phi^3\left(\ln\frac{\phi^2}{\phi_0^2} - 1\right)$ & treat log term as approx const.)

This can be solved as

$$\phi \sim \left(\frac{H}{\lambda}\right)^{\frac{1}{2}} (t_2 - t)^{-\frac{1}{2}}$$



When $\phi \sim \phi_0$ this approx fails; so ϕ increases to some large value ϕ_0 and oscillates around that.



These oscillations, which gradually decay, correspond to a very large number of Higgs particles in a coherent state. You would expect that the Higgs particles would decay into other lighter particles, and they would reheat the Universe almost back up to 10^{15} GeV, but the Higgs field now has an expectation value of ϕ_0, breaking the symmetry. Adjusting so $V(\phi_0)=0$, there is no longer any inflation-driving term; go back to radiation dominated universe with $R \sim t^{1/2}$. The universe now behaves like the standard hot big bang model and baryons can now be produced. Note that they must be produced after inflation or they would be diluted by the exponential expansion and there wouldn't really be any baryons left after this exponential expansion. That is the standard picture of the new inflationary universe. However, there are certain things wrong with this.

Suppose the scalar field is sitting at $\phi=0$. This is a position of equilibrium even though it is unstable equilibrium. Why does the field ever move from this point? People have to wave their hands and talk about quantum fluctuations. A more serious problem is if you're working or to be exponentially expanding now. There will still be a horizon if you take regions far enough apart, but the idea is that all the regions we now see are sufficiently near together - all the points on the intersection of our past light cone with the last scattering surface. If you waited 10^{100} years our past light cone would intersect a much larger part of the last scattering surface and we would see regions that weren't in causal contact. But, for us, there is no longer a horizon problem - it is not surprising that everything is at the same temperature.

Now, the flatness problem: why is Ω so near to 1?
In the hot big bang theory, $\Omega - 1 \propto R^2$
With exponential expansion, $\Omega - 1 \propto R^{-2}$
So, with a long enough period of exponential expansion we can make $\Omega - 1$ so small that even if it grew like R^2 afterwards it would still be small today. The period of exponential expansion Δt multiplied by H must be $H\Delta t > 65$, to agree with observations. It must e-fold at least 65 times, but it doesn't matter if it does more.
If we believe the inflationary universe, $\Omega - 1$ must be very small:

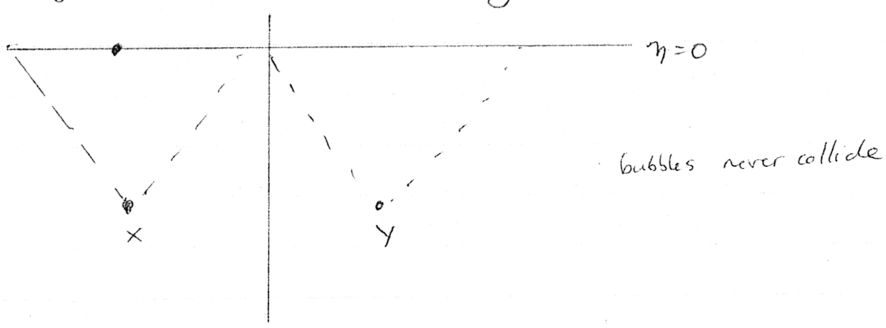
$$\Omega - 1 \sim e^{2(65 - H\Delta t)}$$

so if $H\Delta t$ is 70, $\Omega - 1 \sim e^{-10}$. This solves the flatness problem. There are still difficulties with the smoothness problem, but they are rather different from the problems in the HBB. The monopole problem can be solved too.

To get out of the inflationary era, the original idea was that the scalar field tunnelled through the potential barrier and out to the global minimum of the potential. This can be interpreted as a first order phase transition, just like boiling water, and just as in that case, the way the phase transition occurs is that a bubble of the new phase appears in the surrounding universe. Inside the bubble, $\langle\phi\rangle \neq 0$; outside, $\langle\phi\rangle = 0$. When a bubble like that forms as a result of a quantum fluctuation, it expands outwards, just as in boiling water; eventually it collides with other bubbles which gives a transition to the broken symmetry phase with a nonzero expectation value of ~~$\langle\phi\rangle$~~. ϕ. However, if you've watched water boiling, you'll notice it is very irregular and inhomogeneous. The same occurs on this model of the universe.

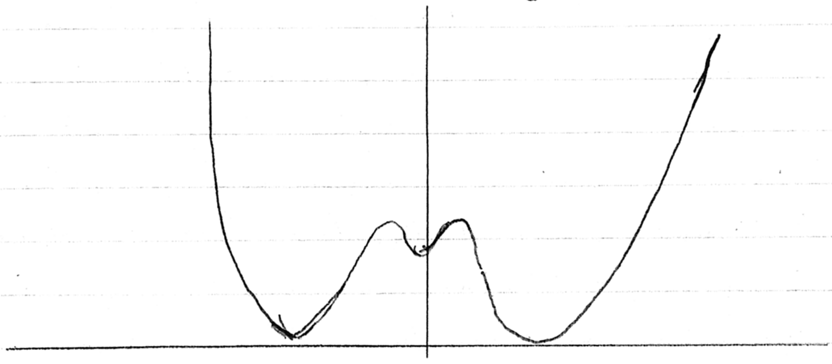
You want to have a long period of exponential expansion to explain the flatness and horizon problems; this means that the rate of formation of these bubbles is very low. If the rate were high, we would get into the broken symmetry phase before much exponential expansion happened.

Suppose the bubbles appear at X and Y, start expanding toward each other at speeds which must be less than c, and in general will never collide. As they expand toward each other, the expansion of the universe is taking them apart even faster and in general they never catch up. This property is because you have an event horizon. The events that a particle can see lie within its past light cone. Because the diagram doesn't go on for ever ($\eta = 0 \Longleftrightarrow t = \infty$), you only ever get a finite amount in the past light cone.



The New Inflationary Universe

What Stephen talked about last time is now called the 'old inflationary model.' There is a Yang-Mills field with a gauge group G and a Higgs field which breaks that group down to a smaller group at lower energies. The Higgs field has an effective potential of the form:



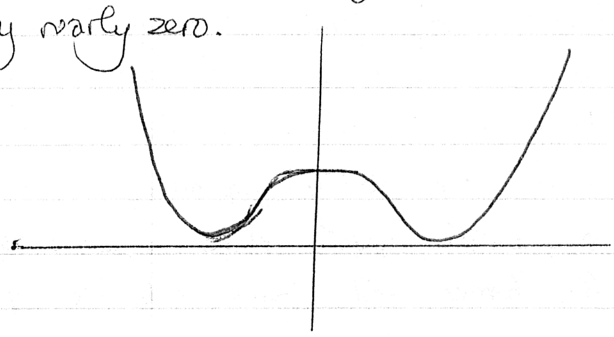
(e.g.)
$$V(\phi) = ce^4\phi^4\left[\ln\left(\frac{\phi^2}{\phi_0^2}\right) - \frac{1}{2}\right] + \phi^2(m^2 + \xi R) + V_0$$

c is a numerical coefficient; e is the Yang-Mills coupling constant. m is the mass of the Higgs field; R is the spacetime curvature and ξ is a renormalization coefficient which may have any value. The ξR term does not appear in flat space, but is neccessary if you want a renormalizable theory in curved space.

The idea is that the field is trapped at $\phi = 0$, with a positive value V_0 of V, and hence a positive energy density. This positive energy density causes the universe to expand exponentially, which removes many of the problems of the hot big bang model. It removes the horizon and flatness problems. Since the universe is not exponentially expanding, it had to get out of this exponential expansion phase in some way. It has to tunnel across the barrier, and forms a bubble of the new phase.

The bubble forms by quantum fluctuation and expands and collides with other bubbles. This is a big problem; de Sitter space is conformal to the lower half (η, r) plane. This diagram only represents the exponentially expanding phase and in fact there will be a BB singularity at the bottom of the diagram. There has to be a long ~~exp~~ inflation period to eliminate the flatness problem so the rate of bubble formation had to be very low; they would not collide because they would form very far apart. Maybe we're inside a single bubble but the trouble is that during inflation, the universe cools to $\left(\frac{10^{15}}{e^{65}}\right)$ GeV which is a very low temperature.

In the bubble formation all the energy $V(0)$ will be released but will not go directly into heating the universe - it will go into accelerating the bubble wall which gets lots of KE. If the bubble wall collides with another one this KE will be released and the energy will be thermalized. But if the bubbles never collide, this never happens and we never get back to the standard hot big bang. The way out of this fatal flaw to old inflation was suggested by Lindé. Suppose the form of V was such that the coefficient of ϕ^2 was zero or very nearly zero.



the effective potential. You should do this not in flat space but in an exponentially expanding universe. As shown in the first lecture an exponentially expanding universe has a Euclidean ~~metric~~ representation which is a 4-sphere, so you should be working out the potential on a 4-sphere. If you assume the potential is flat at $\phi = 0$, you get an infrared divergence in the potential. If you start off by assuming $m = 0$, you find $\exists$ a $\log \phi^2$ term in the potential, so it's not flat at $\phi = 0$ at all, but has an infinite peak!

(no mass $\Rightarrow$ $\exists$ zero mode $\phi = \text{const}$ on 4-sphere; this introduces a log divergence at $\phi = 0$). It doesn't really matter which sign it has, it's really rather bad news.

If you remember the way the effective potential was defined,

$$Z[J] = \int d[\phi]\, e^{-\hat{I}[\phi] - \int dx\, J\phi}$$

over all field configurations ϕ on the 4-sphere

$$\langle \phi \rangle = -\frac{\delta \ln Z}{\delta J},$$

Inverted to find $J(\langle \phi \rangle)$; defined effective action by

$$\Gamma[\langle \phi \rangle] = -\ln Z - \int dx\, J \langle \phi \rangle$$

$$\frac{\delta \Gamma}{\delta \langle \phi \rangle} = -J$$

So the equation of motion of a field in the absence of a source J is

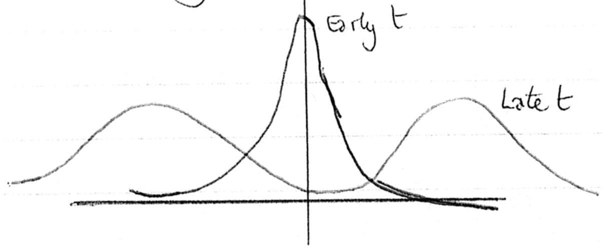
$$\frac{\delta \Gamma}{\delta \langle \phi \rangle} = 0$$

All this was defined in flat space, where it is true that $\langle\phi\rangle$ can be non-zero even when you take the limit $\lim J \to 0$; but if youre working on the 4 sphere that's not true; in fact you find that when $J=0$, always have $\langle\phi\rangle = 0$. The reason for this is that if you have a field configuration ϕ on the 4-sphere, the field configuration $-\phi$ is equally probable when $J=0$ so $\langle\phi\rangle = 0$ always.

You might think that this would be true in flat space as well. However suppose you start with a nonzero value of J everywhere The field configurations $\phi, -\phi$ will not be equally probable - the source term J will make one of them more probable, so $\langle\phi\rangle \neq 0$. Letting $J \to 0$ from a particular direction find in the limit $\langle\phi\rangle$ will still be nonzero - although ϕ & $-\phi$ are equally probable when $J=0$, there is no tunnelling between them. If ϕ has a positive value everywhere in space, ~~so~~ it requires an infinite amount of action to change this to a configuration where ϕ has a -ve value everywhere in space, so there is no tunnelling between the configurations. The reason is that the volume of space is infinite. However, on a 4-sphere the volume of space is finite and you do get tunnelling from ϕ to $-\phi$. This is why $\langle\phi\rangle$ can be nonzero in flat space but must always be zero on the 4 sphere

Another way of putting this:

The probability distn. of ϕ is like this



$\langle\phi\rangle$ always remains zero, but at late times you are either in one well or the other, and there is very little probability of tunnelling between them.

At intermediate times, instead of looking at $\langle\phi\rangle$ it is useful to look at the expectation value of ϕ at two points $\langle\phi(x)\phi(y)\rangle$. Introduce a source K such that the source contribution is

$$-\int dx\, dy\; \phi(x)\, K(x,y)\, \phi(y)$$

instead of $-\int dx\, J\phi$. This enables you to analyse how the scalar field moves away from $\phi = 0$, can go back to previous treatment at late times when field has moved well from $\phi = 0$.

8.11.83

Action is

$$I = \tfrac{1}{2}\int -\phi\,\Box\phi + m^2\phi^2\, dV \quad + \tfrac{1}{2}\int dV dV'\, \phi(x)\, K(x,x')\, \phi(x')$$

$$\equiv -\tfrac{1}{2}\,(\,\underline{\Phi}\,\Delta\,\underline{\Phi} + \underline{\Phi}\,K\,\underline{\Phi}\,)$$

define $\quad A = \Delta + K$

so $\quad A\underline{\Phi}_n = \lambda_n \underline{\Phi}_n$

where $\underline{\Phi}_n$ are functions on S_4, basically hyperspheric harmonics.

$\lambda_n = \ell(\ell+3) + m^2 + K \quad$ if K source is const.

A is an elliptic operator, so the eigenfunctions form a complete orthonormal set on S_4, i.e. any $\underline{\Phi} = \sum a_n \underline{\Phi}_n$ for some a_n.

Hence

$$e^{-W} = \int d[\phi]\; e^{-\frac{1}{2}\underline{\Phi} A \underline{\Phi}}$$

where the measure is

$$d[\phi] = \prod_n \mu \, da_n$$

where the μ makes the dimensions right (ML^{-1})

so

$$e^{-W} = \prod_n \int \mu \, da_n \, e^{-\frac{1}{2}\lambda_n a_n^2}$$

$$= \prod_n \left(\frac{2\pi}{\lambda_n}\right)^{1/2} \mu$$

$$= \prod_n \left(\frac{2\pi\mu^2}{\lambda_n}\right)^{1/2}$$

$$= \left(\frac{1}{2\pi\mu^2}\right)^{-1/2} (\det A)^{-1/2}$$

$$= \left[\det\left(\frac{A}{2\pi\mu^2}\right)\right]^{-1/2}$$

Now $\ln \det A = \operatorname{tr} \ln A$

$\Rightarrow$ $W = \frac{1}{2} \operatorname{tr} \ln (A/2\pi\mu^2)$

$= \frac{1}{2} \operatorname{tr} \ln \; \Delta + K$ $\leftarrow$ $\frac{1}{2\pi\mu^2}$ lost here

$= \frac{1}{2} \operatorname{tr} \ln \; \Delta(1 + \Delta^{-1} K)$

Let $G = \Delta^{-1}$

so $W = -\frac{1}{2} \operatorname{tr} \ln G + \frac{1}{2} \operatorname{tr} \ln (1 + GK)$

Varying W over K $\Rightarrow$

$$\langle \phi(x)\phi(x')\rangle = \frac{\partial \delta W}{\partial K(x,x')} = \rho(x,x')$$

$$= (1 + GK)^{-1} G \qquad \left(= \frac{\partial}{\partial K} \ln(1+GK) \right)$$

Define the effective action (function of the 2 point function ρ)

by a Legendre transformation

$$\Gamma[\rho] = W - \tfrac{1}{2}\operatorname{tr}(K\rho)$$
$$= -\tfrac{1}{2}\operatorname{tr}\ln\rho + \tfrac{1}{2}\operatorname{tr}(G^{-1}\rho)$$

Proof

$$-\operatorname{tr}\ln\rho = -\operatorname{tr}\ln(1+GK)^{-1}G$$
$$= \operatorname{tr}\ln(1+GK) - \operatorname{tr}\ln G$$

so $W = -\tfrac{1}{2}\operatorname{tr}\ln\rho$

$$-\operatorname{tr}(K\rho) = -\operatorname{tr} K(1+GK)^{-1}G = -\operatorname{tr}(1+GK)^{-1}GK$$
$$\operatorname{tr}(G^{-1}\rho) = +\operatorname{tr} G^{-1}(1+GK)^{-1}G = \operatorname{tr}(1+GK)^{-1}GG^{-1} = \operatorname{tr}(1+GK)^{-1}$$

Now $\operatorname{tr}A + \operatorname{tr}B = \operatorname{tr}A+B$

so $\operatorname{tr}A + \operatorname{tr}1-A = \operatorname{tr}1$

so $-\operatorname{tr}(1+GK)^{-1}GK - \operatorname{tr}(1+GK)^{-1} = -\operatorname{tr}1$

ie $-\operatorname{tr}K\rho - \operatorname{tr}G^{-1}\rho = -\operatorname{tr}1$

Hence $-\operatorname{tr}(K\rho) = \operatorname{tr}(G^{-1}\rho) - \operatorname{tr}1$?

<u>so</u>

$\frac{d}{dx}\operatorname{tr}f(x) = f'(x)\frac{d\operatorname{tr}f}{df}$

$$-2\Gamma = \operatorname{tr}\ln\rho - \operatorname{tr}G^{-1}\rho$$
$$= \ln\det\rho - \operatorname{tr}G^{-1}\rho$$
$$-2\frac{\delta\Gamma}{\delta\rho} = \rho^{-1}\operatorname{tr}'(\ln\rho) - G^{-1}\operatorname{tr}'(G^{-1}\rho)$$
$$=? \quad -\rho^{-1} + G^{-1}$$

but $-2\frac{\delta\Gamma}{\delta\rho} = K$

so if $K=0$,

$$\rho^{-1} = G^{-1}$$
$$G^{-1}\rho = 1 = \delta(x,x')$$

ie $\Delta\rho = \delta(x,x')$

so

$$(-\Box_x + m^2)\rho(x,x') = \delta(x,x')$$

This is the eqn of evolution of ρ if no sources are present

Boundary conditions:

In the new inflationary scenario, in the early universe, the very high temperature kept the expectation value of the field about zero : $\langle\phi\rangle = 0$, $\langle\phi^2\rangle \sim O(T^2)$

As the Universe expanded, the temperature dropped so $\langle\phi^2\rangle$ was small : the field is concentrated at $\phi = 0$.

We want to solve with ρ small at time $t = t_0$ when the universe began its exponential expansion..

At $t > t_0$, metric is like that of de Sitter space $\rightarrow$ consider solutions on S_4. If $m^2 > 0$, there is a unique solution with ρ_E regular on a 4-sphere - this is why we use Euclidean space, there is a unique green's fn! ; but here, if $m \to 0$, the Green's function diverges on Euclidean space

$$-\Box\rho_E = \delta$$

$$\Rightarrow \quad \underset{\substack{\parallel \\ 0 \text{ since no bdy}}}{-\int \Box\rho_E \, dV} = \int \delta \, dV = \frac{8\pi^2}{3} H^{-4}$$

Define a new Green's function $\rho_0(x', x)$

$$(-\Box_x + m^2)\rho_0(x, x') = \delta(x, x') - \frac{3H^4}{8\pi^2}$$

This is well behaved as $m \to 0$. Analytically continue it back to Lorentzian de Sitter space with asymptotic behaviour

$$\rho_0(x, x') \sim \begin{cases} (2\pi^2 |x - x'|^2)^{-1} & |x - x'| \ll H^{-1} \\ -\frac{1}{4\pi^2} H^2 \ln(H|x - x'|) & |x - x'| \gg \underset{\substack{\uparrow \\ \text{event horizon length}}}{H^{-1}} \end{cases}$$

The solution on a 4 sphere is

$$\rho(x, x') = \rho_0(x, x') + \rho_1(x, x') + \rho_2(x, x')$$

Where the ρ_1 is function only of the t coord of x, x'

For $m^2 = 0$, $\rho_1(x, x') = \frac{H^3}{8\pi^2}(t_x + t_{x'} - 2t_0)$

Back into eqn for ρ:

$$(-\Box_x + m^2)\rho_2(x, x') = 0$$

so ρ_2 is small at $t = t_0$. Solving in de Sitter space, $\rho_2 \to \text{const} \sim H^2$

Ignoring this const,

$$\rho(x, x') = \rho_0 + \rho_1 + \text{const}$$

To find $\langle\phi^2\rangle$, want $x = x'$ but the Greens fn diverges.

We must subtract out the divergences

$$\langle \Phi^2 \rangle = \frac{H^3}{4\pi^2}(t-t_0)$$

??

Flat region of potential,

but when $\langle \phi^2 \rangle \approx \frac{H^2}{\beta^4}$,

the curvature is significant, and the expectation value of Φ^2 increases more rapidly, but then tunnelling between ϕ and $-\phi$ is very small, so may return to original method in terms of Φ as described in the previous lecture - reaches a min. and oscillates.

Variations in the rate of run-down cause fluctuations possibly leading to galaxy formation.

Quantum Gravity

20 years ago most people believed that there would not have been a singularity, but a previous contracting phase and somehow there would be a bounce and it would expand again. But Hawking and Penrose proved some singularity theorems that showed this could not happen in classical GR and that you inevitably got a singularity. A singularity is a very bad thing because the theory breaks down - you cannot formulate the Einstein field equations at the singularity and so there is no way of predicting what comes out of the big bang. For some people that's an attraction, because they'd say that its not up to science to say how the universe began; God could start off the universe in any way he wanted. That may be true, but in that case he could also have made it evolve in an arbitrary way; it doesnt seem to be that way. There are laws, so it is not unreasonable to say that there are laws on how it started. In order to understand these laws we have to quantize gravity. This is a situation like the classical model of the atom at the beginning of the century. According to classical physics, the electron would radiate energy, spiralling into the nucleus. The atom and all matter would collapse - rather like the collapse of the Universe in classical GR. Stephen will show that quantum theory saves you here too.

Stephen will adopt the Euclidean approach to quantum gravity ; the basic quantities in the theory to be positive definite metrics $g^{\mu\nu}$, the probability of having $g^{\mu\nu}$ and some matter field ϕ is

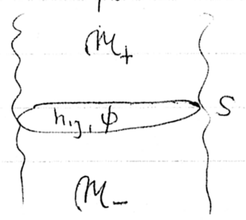
$$P \propto e^{-\hat{I}[g_{\mu\nu}, \phi]}$$

where the Euclidean action $\hat{I}$ is

$$\hat{I} = \frac{m_p^2}{16\pi}\left(-\int_{\partial M} 2K h^{\frac{1}{2}} d_3 x - \int_{M} \left(R - 2\Lambda - \frac{16\pi}{m_p^2} L\right) g^{\frac{1}{2}} d_4 x\right)$$

The $R-2\Lambda$ term is the normal gravitational action, and in most of what Stephen will say $\Lambda = 0$. Most people would say that's all we have in the action, but curvature scalar R contains second derivatives of the metric linearly; these can be removed by integrating by parts which creates a boundary term which is cancelled out by the surface term included in the action.

In most situations don't want to know the probability of the 4 - metric we want to know the probability of a more restricted class of observables. We can derive such a probability from this expression by integrating out all the things you don't observe. In cosmology you're interested in observables in a *finite region*: eg if you want the prob. of finding a 3-D submanifold S: S divides M into 3 parts



S has the induced metr h_{ij} and matter configuration ϕ

$$P(h_{ij}, \phi) = \int_C d[g_{\mu\nu}]\, d[\phi]\, e^{-\hat{I}[g_{\mu\nu}, \phi]}$$

The $\int$ is over all 4-metrics containing the submanifold S with metric h_{ij} & matter ϕ. In fact only choose some class C of 4-manifolds and metrics $g^{\mu\nu}$, & only consider probabilities from members of that class. The choice of that class determines the quantum state of the system. Stephen will say more about the choice of S later.

We fix the metric on the 3-surface, but allow it to vary in M^+ and M^-. We can factorize the probability

$$P = \Psi_+(h, \phi)\Psi_-(h, \phi)$$

Ψ_+ is a path $\int$ on all metrics ~~in M_+~~ just above S
Ψ_- - - - - - - - - - - - - - - - below S.

This divides the prob^y into product of 2 quantities which Stephen will call amplitudes or wave functions. In fact in many cases Ψ_+ and Ψ_- will be real and equal. These wave functions Ψ_+ and Ψ_- contain all the information about the quantum state of the universe. They obey a functional d.e. which is very like the Schrödinger equation with zero eigenvalue:

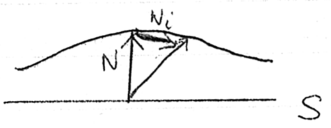
$$H\Psi_\pm = 0$$

where H is a certain functional operator. Stephen will now derive this equation.

Introduce a time coordinate τ, with $\tau = 0$ on S.

Write $\quad ds^2 = (N^2 + N_i N^i)\, d\tau^2 + {}^3g_{ij}\, dx^i dx^j \quad$ near S

The quantity N is the lapse function; it measures the distance you have to go along the normal to S to get to the next surface of constant τ.
There are 3 coordinates $x_i \in S$, and 3 factors N_i, the shift factor, telling you how must move sideways relative to the normal to stay at the same values of x_i. This is just a convenient way of writing the metric. You can write the action as



$$\hat{I}[g,\phi] = -\frac{m_p^2}{16\pi} \int d_3x\, d\tau \; N h^{1/2} \left[K_{ij}K^{ij} - K^2 + {}^3R(h_{ij}) - 2\Lambda + \frac{16\pi}{m_p^2} L[\phi] \right]$$

$$\psi[h,\phi] = \int_C d[g]\, d[\phi]\, e^{-\hat{I}[g,\phi]}$$

The wave function does not depend on time.
If you move along the normals to S, it can't change the wave function. Functionally differentiate both sides wrt N:

$$\frac{\delta \psi}{\delta N} = 0$$

$$= -\int d[\phi]\, d[g]\, \frac{\delta \hat{I}}{\delta N}\, e^{-\hat{I}[g,\phi]}$$

so would want roughly

$$H = -h^{1/2}\left(K_{ij}K^{ij} - K^2 + {}^3R(h_{ij}) - 2\Lambda + \frac{16\pi}{m_p^2} L \right) = 0$$

since this doesnt ~~app~~ include N dependence, except for matter Lagrangian.
In fact must replace L by $\frac{1}{2}T_{nn}$, the normal component of the stress tensor.

$$H = -h^{\frac{1}{2}}\left(K_{ij}K^{ij} - K^2 + {}^3R - 2\Lambda + \frac{16\pi}{2m_p^2} T_{nn}(\phi)\right) = 0$$

K_{ij} is the 2^{nd} fundamental form and measures the way that S is embedded in the 4-geometry.

$$K_{ij} = \frac{1}{N}\left\{ \frac{1}{2}\frac{\partial h_{ij}}{\partial \tau} + N_{(i|j)} \right\}$$

It can be expressed in terms of a functional derivative of the action $\hat{I}$ wrt 3-metric h_{ij}

$$\frac{\delta \hat{I}}{\delta h_{ij}} = \frac{m_p^2}{16\pi h^{\frac{1}{2}}} \left(K_{ij} - h_{ij}K \right)$$

Classically, $H = 0$, but now $H = \frac{\delta \hat{I}}{\delta N}$. Substitute into path integral, and definition of $\Psi[h,\phi]$. Get:

$$\left[- G_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} - h^{\frac{1}{2}}\left({}^3R + 2\Lambda + \frac{8\pi}{m_p^2} T_{nn}\left(\frac{\delta}{\delta\phi}, \phi\right)\right)\right]\Psi = 0$$

Wheeler-de Witt equation

where

$$G_{ijkl} = \frac{h^{\frac{1}{2}}}{2}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl} \right)$$

is the metric on superspace, the space of all 3-metrics.
Forget about ϕ - suppose only have pure gravity.
The fact that the wave function does not depend on time explicitly means that a small displacement in space and time will lead to a constraint the ~~equation~~ wave fn. must satisfy so the wave function doesnt change. One way of looking at

this equation is that it's just a constraint the wave function must satisfy to be time independent. Another is to look at it as the analogue of the Schrödinger Equation; the right hand side is zero since Ψ does not depend on time. All we have to do is solve this Wheeler-de Witt equation and that will describe everything in the Universe; but that's easier said than done. ~~We~~

We can regard this object G_{ijkl} as G_{AB} where $ij \to A$ $kl \to B$, and A, B take 6 values, due to the symmetries of G. So $-G$ is a 6×6 matrix with signature

$$\mathrm{sig}(-G) = + - - - - -.$$

This metric multiplies the 2nd functional derivatives wrt the 3 metric. So the WDW eqn can be thought of as a hyperbolic equation on the space of 3-geometries with $+$ for time dimension and $-$ for the 5 space dimensions. We choose the time coordinate to be the volume factor $h^{1/2}$. So we can study the wave function and its derivative wrt $h^{1/2}$ on one surface of constant $h^{1/2}$; solve this Cauchy problem, find Ψ for other values of $h^{1/2}$. But the coordinate has a restricted range:

$$h^{1/2} > 0$$

The problem of solving WDW is the same as knowing what happens to Ψ at small $h^{1/2}$, the same as knowing what happens at very early times in the Universe. Next lecture, Stephen will give a prescription for boundary conditions at small $h^{1/2}$.

It is not always convienient to work with a variabl like $h^{1/2}$ with a restricted range; the canonically conjugate quantity is $K = \mathrm{tr}\, K_{ij}$. One can define a wave function $\qquad \Phi = \Phi(\tilde{h}_{ij}, K, \phi)$

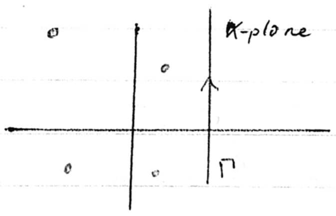
where $\tilde{h}_{ij}$ is h_{ij} up to a confomal factor.

$$\Phi[\tilde{h}_{ij}, K, \phi] = \int_C d[g]\,d[\phi]\, e^{-\hat{I}^k[g,\phi]}$$

This wave function Φ is given by a similar path integral with a slightly different action; the surface term is different.
Φ is related to Ψ by a Laplace transform

$$\Phi = \int_0^\infty d[h^{1/2}] \exp\left(-\frac{m_p^2}{12\pi}\int d_3x\, h^{1/2}K\right) \Psi[h_{ij}, \phi]$$

$$\Psi[h_{ij},\phi] = \int_\Gamma d\left[\frac{m_p^2}{24\pi i} K\right] \exp\left(\frac{m_p^2}{12\pi}\int d_3x\, h^{1/2}K\right) \Phi[\tilde{h}_{ij}, K, \phi]$$



$\int$ contour Γ for each x.

If $h^{1/2} < 0$, can close contour in right half plane and get zero.
So if you define Φ in this way it is always zero when $h^{1/2} < 0$ which is what we want; Ψ is related to the prob.^{y} of finding a ~~given~~ 3-metric with a given 4-geometry which is zero when $h^{1/2} < 0$.

Boundary conditions: the possibilities are most naturally

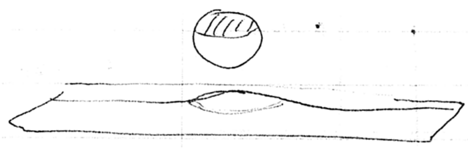
① Compact metrics with matter fields regular on them.

② Non-compact metrics which are asymptotically Euclidean or Anti-de Sitter, matter → 0 at ∞
'AE'

Problems are with ②: have only ~~region~~ observations in a finite region of spacetime and not at ∞.

AE metric

Disconnected metric - compact and AE parts

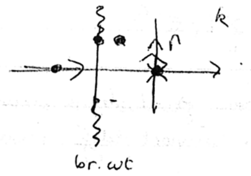


This can be approximated by a connected metric, with a tube contributing negligible action joining the parts. So such disconnected metrics cannot be excluded.

So as ② is dominated by ①-like contributions, its more natural to choose ①. Entire universe is compact, there's no outside disconnected region. (doesn't leave much room for God) related by a LT

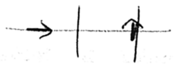
$$\psi(a) = \frac{N_k}{2\pi i}\int_\Gamma dk\, \exp(ka^3 - \hat{I}^k)$$

Where the contour Γ is // im k axis

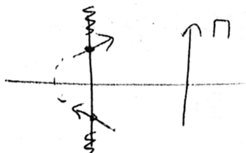


Use stationary phase;
If $Ha < 1$, Q real

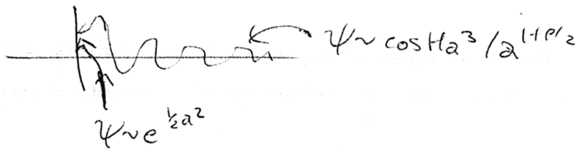
Dominant term is − not +, can pass thru ✱ and follow line of steepest descent for − case.



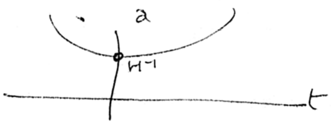
If $Ha > 1$, sta phase points are on im axis



Can deform contour to pass thru both along line of steepest descent
So wave function solution oscillates in a.



To interpret all this consider classical Lorentzian soln with Λ term (de Sitter space)
$a = H^{-1}\cosh Ht$



Min. radius is H^{-1}; $aH \geq 1$ ⇒ expect damped for $a < H^{-1}$ in qm.

$$\cos \to \exp i\,\frac{Ha^3}{3} + \exp - i\,\frac{Ha^3}{3}$$

Associate one term with expanding and one with contracting phase.

$$K\psi = \frac{\partial}{\partial a^3}\psi \qquad \psi_{osc} \Rightarrow K^2 < 0 \Rightarrow \underset{\text{Eucl.}}{K_E} = \underset{\text{Loren.}}{iK_L}$$

$$K^2 \Leftrightarrow \frac{\partial}{\partial a^3}\frac{\partial}{\partial a^3}$$

The proposition is that the quantum state of the Universe is defined by a path integral over compact metrics. This can be tested by predictions it makes.

WdW equation, with Ψ given by the semiclassical approximation to path integral over compact metrics

$$\Psi \sim N_0 \sum_i A_i e^{-B_i}$$

A_i is the det. of small fluctuations around solutions
B_i is the action on compact ~~tdef~~ metric which is a soln of the Euclidean EFE.

The WdW is wave equation on ∞-dimensional space. This makes it hard to solve. We instead truncate ∞ degrees of freedom of the gravitational field, to finite number − one. This is 'minisuperspace'.

Consider a spatially closed homogeneous universe of radius a.

$$ds^2 = \sigma^2 \left(d\tau^2 + a^2(\tau)\, d\Omega_3^2 \right), \qquad \sigma^2 = \frac{2}{3\pi M_p^2}$$

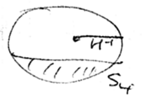
If no matter field, $\Lambda = \frac{3\lambda}{\sigma^2}$

WdW:

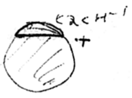
$$\frac{1}{2}\left[\frac{1}{a^p}\frac{\partial}{\partial a} a^p \frac{\partial}{\partial a} - a^2 + \lambda a^4 \right]\Psi(a) = 0$$

The p comes from ambiguity in factor ordering in WdW, equivalent to ambiguity in measuring path integral. Doesn't depend on value of p

The solution of EFE on 4-sphere of radius H^{-1};

− 

$H^2 = \lambda$



Get 2 possibilities,

$$\hat{I}_{\pm}(a) = \frac{-1}{3H^2}\left(1 \pm (1 - H^2 a^2)^{3/2}\right)$$

If $a > H^{-1}$, no real solution.
Use different representation (analogous to momentum rep)

$$\hat{I}^{k} = -\frac{1}{3H^2}\left(1 - \frac{3k}{(9k^2 + H^2)^{1/2}}\right)$$

$K^2 < 0 \Rightarrow$ real Lorentzian K $\Rightarrow$ Lor. 4geom

Ψ expl, $K^2 > 0 \Rightarrow K_E \in \mathbb{R}$ $\Rightarrow$ Eucl. 4geom

$(\Lambda > 0)$.

S. W. Hawking
Nov 17, 1983

The Cosmological Constant is Probably Zero

In last May's seminar, Stephen suggested a possible solution to the cosmological constant problem based on $n=8$ supergravity and topological fluctuations. Stephen later realized that the use of $n=8$ supergravity was not essential and that he could do it in a much simpler, almost trivial way.

It seems that the universe is described very well by the Robertson-Walker model

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1+kr^2} + r^2 d\Omega_2^2\right]$$

By observing the magnitude and redshift of distant galaxies, we can determine 2 quantities observationally: H_0 and q_0. In practice, only know H_0 to a factor 2 and we don't know q_0 at all, we can only put bounds on it: $|q_0| < 5$ is fairly safe.

From the Einstein field equations, we can then put an upper bound on the cosmological constant; In units where

$$c = \hbar = Gm_p^2 = 1,$$

$$\frac{|\Lambda|}{m_p^2} \lesssim 10^{-120}$$

Observational limits on the mass of the photon give $m_\gamma^2/m_p^2 \leq 10^{-48}$ so Λ in this sense is the physical quantity which is most accurately known to be zero. The photon is believed to be of exactly zero mass from this result - in fact we invent a symmetry called gauge invariance which makes its mass exactly zero. There is no similar symmetry which makes Λ exactly zero. You might say that maybe we should just assume that Λ is exactly zero. But the effective value of Λ changes every time you have spontaneous symmetry breaking. SSB at an energy μ gives $\Lambda_{eff} \sim \mu^4/m_p^2$, and there appear to be many broken symmetries in the universe. – EW symmetry has $\mu \approx 100$ GeV, chiral symmetry 100 MeV, GUTS, etc. But even with these two get changes in Λ violating these upper limits by at least 40 orders of magnitude. It is very difficult to believe that the original Λ was adjusted so precisely that after all these SB's you end up with a Λ less than the upper bound quoted.

What Stephen wants to suggest today is really a very simple idea – Λ is not neecessarily zero, in fact Λ can take all possible values, but the probability of having a very small value is much higher than having a large value. This is what Stephen means by "Λ is 'probably' zero". We need a way to have a Λ_{eff} with a whole range of values. One way of doing this would be to include the cosmological constant in the path integral. Another much more natural looking way is if you have a 3 index antisymmetric tensor field,

$$A_{\mu\nu\rho} = A_{[\mu\nu\rho]}$$

such as arises in some forms of dimensional reduction in supergravity, but you can have them anyway without supergravity. $A_{\mu\nu\rho}$ is very like the electromagnetic potential but has a couple more indices.

$$F_{\mu\nu\sigma\rho} = \nabla_{[\mu} A_{\nu\rho\sigma]}$$

F is invariant under gauge transformations of the type

$$A_{\mu\nu\rho} \rightarrow A_{\mu\nu\rho} + \nabla_{[\mu} C_{\nu\rho]}$$

Action is of the form

$$I' = \int F^2 \sqrt{g}\, d_4x$$

$$\frac{\delta I}{\delta A} = 0 \quad \text{leads to the analog of the Maxwell equation,}$$

$$\nabla^{\mu} F_{\mu\nu\rho\sigma} = 0$$

Now since F is completely antisymmetric,

$$F_{\mu\nu\rho\sigma} = \phi \, \epsilon_{\mu\nu\rho\sigma}$$

with $\nabla^\mu \phi = 0$

So ϕ is just a constant whose value is undetermined.
The action is then

$$\int \phi^2 \sqrt{g} \, d_4 x$$

which behaves like a cosmological constant $\Lambda = \phi^2$,

$$G^{\mu\nu} + \tfrac{1}{2}\phi^2 g^{\mu\nu} = 0$$

Contributions from other symmetry breakings are of any magnitude and sign; Λ_{eff} = arbitrary.
Another way of getting a Λ_{eff}; start with $n=8$ supergravity. At a critical value of the coupling constant there will be a phase transition from a spacetime which is basically smooth anti de Sitter space to a spacetime which had a lot of topological fluctuations per unit volume on the scale of the Planck length. Looking on a larger scale you wouldn't see these fluctuations, instead this spacetime would appear to be smooth. But these fluctuations would contribute to the effective Ricci tensor

$$R^{\mu\nu}_{eff} \sim \rho^{1/2} g^{\mu\nu}$$

where ρ is the density of topological fluctuations per unit volume, which is arbitrary. There are other ways you can get an effective Λ of arbitrary magnitude. Stephen will assume we have such a Λ_{eff}.
Stephen will use the Euclidean approach to QG where

$$P(+\text{def } 4\text{-metric } g^{\mu\nu}) = e^{-\hat{I}[g^{\mu\nu}]}$$

We expect metrics near to the solutions of the classical field equations.

If $\Lambda_e > 0$, then a solution to the EFE is necessarily compact and the action is bounded below by the action on a 4-sphere, the solution of the greatest symmetry,

$$\hat{I} = -\frac{3\pi m_p^2}{\Lambda_e}$$

If $\Lambda_e < 0$, can be either compact or non compact

If it is compact, $\hat{I} > 0$

If it is non-compact, $\hat{I}$ is +ve and infinite $\hat{I} = +\infty$

Probability:

$$P < e^{3\pi m_p^2/\Lambda_e}$$

So P is higher for smaller Λ_e.

In the limit $\Lambda_e \to 0$, P becomes ∞ high. This is only the relative probability, you have to normalize by some factor. You see that metrics with $\Lambda_e = 0$ are infinitely more probable than any metric with $\Lambda_e \neq 0$ and thats why the measured value of Λ is so small.

Questions:

In the early universe the matter fields are not in their ground state, so can have a Λ_{eff} of any value. Nowadays the volume of spacetime is very large so Λ_{eff} is now very important - there is a $\int \Lambda \, dV$ in the action; early on the volume was small so Λ_{eff} didnt affect the probability. The probability of a completely empty universe with $\Lambda_e = 0$ is very much higher but we're interested in the ones with some matter in.

Should P be so big if $\Lambda = 0$? We only have an upper bound; if it were as high as that limit, ST would be empty de Sitter space which it, in ly is not. The prob^y of solutions with matter still gives a very much higher prob^y with Λ small, but not exponentially higher. It's more like $1/\Lambda$ which is still quite big when Λ is small. $\Lambda = 0$ is a rather singular case; study $\Lambda \to 0^+$.